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Gauge principle for flows of an ideal fluid

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Abstract

A gauge principle is applied to flows of a compressible ideal fluid. First, a free-field Lagrangian is defined with a constraint condition of continuity equation. The Lagrangian is invariant with respect to global $SO(3)$ gauge transformations as well as Galilei transformation. From the variational principle, we obtain the equation of motion for a potential flow. Next, in order to satisfy local $SO(3)$ gauge invariance, we define a gauge field and a gauge-covariant derivative. Requiring the covariant derivative to be Galilei-invariant, it is found that the gauge field coincides with the vorticity and the covariant derivative is the material derivative for the velocity. Based on the gauge principle and the gauge-covariant derivative, the Euler's equation of motion is derived for a homentropic rotational flow. Noether's law associated with global $SO(3)$ gauge invariance leads to the conservation of total angular momentum. This provides a gauge-theoretical ground for analogy between acoustic-wave and vortex interaction in fluid dynamics and the electron-wave and magnetic-field interaction in quantum electrodynamics.

Keywords: Ideal fluid; Gauge theory; $SO(3)$; Vorticity; Euler equation

1. Introduction

A guiding principle of the gauge theory is that laws of physics should be expressed in a form that is independent of any particular coordinate system. Typical examples of its successful application are the Dirac equation and the Yang-Mills equation in the quantum field theory.

Study of fluid flows is considered as a field theory in Newtonian mechanics, more precisely, a field theory of mass flows which are invariant under the Galilei transformation. It is generally accepted that investigation of vorticity dynamics is essential for full understanding of fluid

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motions. On the other hand, there are various similarities between fluid dynamics and electromagnetism. One representative example is the law between the velocity field and vorticity field which is equivalent to the Biot–Savart law in electromagnetism between the magnetic field and electric current. Another is the scattering of acoustic (or water) waves by a tubular vortex (Kambe and Mya Oo, 1981; Berry et al., 1980; Umeki and Lund 1997; Coste et al., 1999), which is analogous to the interaction of electron-waves impinging on a tubular magnetic field (Peshkin and Tonomura, 1989). One may ask whether the similarities are merely an analogy, or have a solid theoretical background.

Let us consider the scenario of the gauge principle in the quantum field theory (Frankel, 1997; Quigg, 1983). First, a free-particle Lagrangian is defined in such a way as invariant under the Lorenz transformation. Next, a gauge principle is applied to the Lagrangian, requiring it to have a *symmetry*, namely, global and local gauge invariance. Thus, a gauge field such as the electromagnetic field is introduced to satisfy the *local* gauge invariance for which the gauge group is the group $U(1)$, whereas the relevant group for the Yang-Mills field is the Lie group $SU(2)$. In the present problem, the gauge group relevant to fluid flows is considered to be the rotation group $SO(3)$.

A gauge theory of rotation invariant Lagrangian with an internal $O(3)$-symmetry was developed for the Bohr model of nuclear collective rotation of a finite number of modes (Fujikawa and Ui, 1986). There is a similarity between this system (of five field variables) and the fluid flows (of infinite dimensions). In particular, both systems are considered a dynamical system, in other words, so-called $d=1$ field theory in the sense that the gauge field is defined for the covariant derivative of *time* evolution only. The gauge field of the nuclear collective rotation was found to be the angular velocity.

In the present paper, we seek a scenario in a fluid flow which has a formal equivalence with the gauge theory. In order to go further, we define a Galilei-invariant free-field Lagrangian for fluid flows and examine whether it has global and local gauge invariance. It will be found below that the gauge field of a fluid flow coincides with the vorticity.

2. Free-field Lagrangian of fluid flows

Suppose that a free-field Lagrangian of fluid flows in a 3D subdomain $M^3 \subset \mathbb{R}^3$ is given by

$$\Lambda_{\mathrm{f}}[\boldsymbol{v}, \rho, \phi] = \int_{M^3} \mathrm{d}^3x \left[\rho(x)\left(\frac{1}{2}\langle \boldsymbol{v}, \boldsymbol{v}\rangle - \varepsilon(\rho)\right)\right] + \int_{M^3} \mathrm{d}^3x\, \phi(x)\left[\partial_t\rho + \mathrm{div}(\rho\boldsymbol{v})\right], \tag{1}$$

(Herivel, 1955; Serrin, 1959), where $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = (v^1)^2 + (v^2)^2 + (v^3)^2$ is a scalar product of a velocity field $\boldsymbol{v}(x,t) = (v^i)$, ρ the fluid density, $\varepsilon(\rho)$ the internal energy per unit mass, $\phi(x)$ a scalar function acting as a Lagrange multiplier, and $\partial_t = \partial/\partial t$, with $x \in M^3$ and t the time. The fluid is assumed to be *homentropic*, i.e. the specific entropy s is uniform in space. It is shown in Appendix A that the first term in the Lagrangian Λ_{f} is regarded as invariant with respect to the Galilei transformation according to the prescription that it is replaced by $\Lambda_{\mathrm{L}}^{(0)}\,\mathrm{d}t$ of (15) when necessary.

The action principle for a fluid flow is given by $\delta\mathscr{A} = 0$, where $\mathscr{A} = \int_{t_1}^{t_2} \Lambda_{\mathrm{f}}[\boldsymbol{v}, \rho, \phi]\,\mathrm{d}t$ with a fixed condition at both ends of the t integration. Independent variations are taken for the Eulerian variables: ϕ, $\boldsymbol{v}$ and ρ. The variational principle $\delta\mathscr{A} = 0$ for independent arbitrary variations $\delta\boldsymbol{v}$, $\delta\phi$

and $\delta\rho$ results in the following three expressions:

$$\boldsymbol{v} = \text{grad}\,\phi, \quad \partial_t\rho + \text{div}(\rho\boldsymbol{v}) = 0, \quad \tfrac{1}{2}v^2 - h - \partial_t\phi - \boldsymbol{v}\cdot\text{grad}\,\phi = 0, \tag{2}$$

respectively (Serrin, 1959), where $h = \varepsilon + \rho\,\mathrm{d}\varepsilon/\mathrm{d}\rho = \varepsilon + p/\rho$ (since $\mathrm{d}\varepsilon/\mathrm{d}\rho = p/\rho^2$ with s fixed) is the specific enthalpy. Note that $\mathrm{d}h = (1/\rho)\,\mathrm{d}p$ with s fixed (p: the pressure). The first equation represents that the velocity $\boldsymbol{v}$ has a potential ϕ. The second is just the continuity equation for a compressible fluid. The third equation corresponds to an integral of the equation of motion. In fact, applying 'grad' to the third of (2) and using $\boldsymbol{v} = \text{grad}\,\phi$, we obtain the Euler's equation of motion for a potential flow of an ideal fluid

$$\partial_t\boldsymbol{v} + \text{grad}(\tfrac{1}{2}v^2) = -\text{grad}\,h, \quad \text{where grad}\,h = \frac{1}{\rho}\,\text{grad}\,p. \tag{3}$$

It can be shown that both of the left-hand side of (3), defined by

$$\mathrm{D}_t\boldsymbol{v} := \partial_t\boldsymbol{v} + \text{grad}(\tfrac{1}{2}v^2), \tag{4}$$

and its right-hand side are Galilei-invariant, respectively.

The velocity field $\boldsymbol{v}(x,t)$ thus obtained is *irrotational* since the vorticity is given by $\text{curl}\,\boldsymbol{v} = \text{curl}(\text{grad}\,\phi) = 0$. Regarding the irrotational field, the following observation would be of some significance. It is well-known that a flow of a superfluid is described by a velocity potential in the ground state and the motion is governed by the equation of an ideal fluid in the form of (3) for a homentropic fluid (Landau and Lifshitz, 1987, Chapter XVI). Therefore, the superfluid flow is irrotational. The *vorticity* signifies rotational motion of fluid, i.e. *local rotation* of fluid material. However, if particles are equivalent and indistinguishable such as the superfluid He^4, the rotational motion would not be captured.[1] Therefore the flow would be irrotational. This is not the case when we consider the motion of a fluid composed of distinguishable particles such as in an ordinary fluid. Local rotation is distinguishable and there must be a formal structure to take into account the local rotation of fluid particles. This is considered in the next section.

3. Gauge transformation and covariant derivative

In the gauge principle, a free-field Lagrangian Λ_f is required to be invariant with respect to a global gauge transformation.[2] Regarding the gauge transformation of fluid flows, the relevant structure group is the rotation group $SO(3)$.[3] This is interpreted as follows. Consider a transformation of a vector, $\boldsymbol{v} \mapsto \boldsymbol{v}' = R\,\boldsymbol{v}$ with an element R of the group $SO(3)$. Then the scalar product $\langle\boldsymbol{v},\boldsymbol{v}\rangle$ is invariant with respect to the transformation $\boldsymbol{v} \mapsto \boldsymbol{v}'$, i.e. $\langle\boldsymbol{v}',\boldsymbol{v}'\rangle = \langle R\boldsymbol{v}, R\boldsymbol{v}\rangle = \langle\boldsymbol{v},\boldsymbol{v}\rangle$. Thus, the phase transformation in the quantum theory is replaced by an *isometry* transformation of the rotation group $SO(3)$.

If a proposed Lagrangian is invariant under a global gauge transformation (with a fixed R) as well as the Galilei transformation, then the gauge principle demands that a partial derivative ∂ (if

[1] The superfluid He^4 obeys Bose–Einstein statistics in which particles are equivalent and indistinguishable.

[2] In the quantum electrodynamics, this requires that its form is invariant under a phase transformation of the wave function, $\psi \mapsto e^{i\alpha}\psi$, where the phase α is a real constant. This keeps the probability density $|\psi|^2$ unchanged. The global gauge invariance results in conservation of Noether current (Frankel, 1997; Quigg, 1983).

[3] $SO(3)$ is the group of all the special linear orthogonal transformations of $\mathbb{R}^3$, characterized by $\det R = 1$.

any) is to be replaced with a *covariant derivative* including a gauge field $\Omega(x), \partial \to \nabla = \partial + \Omega(x)$, so that the derivative ∇ acquires a local gauge invariance.

With respect to the local gauge transformation, we write the transformation at a point x as $R(x) = \exp[\theta(x)] = I + \theta + O(|\theta|^2)$, where I is the unit matrix, $R \in SO(3)$ and θ is an element of the Lie algebra $\mathbf{so}(3)$. The θ is represented by a skew-symmetric 3×3 matrix. We consider an infinitesimal transformation for which $|\theta| \ll 1$. Then, the velocity vector $\boldsymbol{v}$ is transformed as

$$\boldsymbol{v}(x) \to \boldsymbol{v}'(x) = R(x)\boldsymbol{v}(x) \approx \boldsymbol{v} + \theta\boldsymbol{v} = \boldsymbol{v} + \hat{\theta} \times \boldsymbol{v}. \tag{5}$$

up to the first order of $|\theta|$, where the matrix multiplication $\theta\boldsymbol{v}$ is replaced by an equivalent form of a vector product $\hat{\theta} \times \boldsymbol{v}$ by using an axial vector $\hat{\theta}$ associated with the skew-symmetric matrix $\theta \in \mathbf{so}(3)$. The scalar product is invariant under the local transformation: $\langle \boldsymbol{v}', \boldsymbol{v}' \rangle(x) = \langle \boldsymbol{v}, \boldsymbol{v} \rangle(x)$.

It is remarkable that the transformed field $\boldsymbol{v}'(x) = R\boldsymbol{v}$ is *rotational*, even if $\boldsymbol{v}$ is irrotational. In fact, one can represent the second term of (5) as $\hat{\theta} \times \boldsymbol{v} = \mathrm{curl}\, B + \mathrm{grad}\, f$ by using a vector potential B and a scalar potential f, together with the gauge condition $\mathrm{div}\, B = 0$. Taking curl of $\hat{\theta} \times \boldsymbol{v}$, one obtains

$$\mathrm{curl}(\hat{\theta} \times \boldsymbol{v}) = \mathrm{curl}(\mathrm{curl}\, B) = -\Delta B,$$

where Δ is the Laplacian. The vector potential B is determined by the equation, $\Delta B = (\hat{\theta} \cdot \mathrm{grad})\boldsymbol{v} + (\mathrm{div}\, \hat{\theta})\boldsymbol{v} - (\boldsymbol{v} \cdot \mathrm{grad})\hat{\theta} - (\mathrm{div}\, \boldsymbol{v})\hat{\theta}$. Thus, it is found that the gauge transformation introduces a *rotational* component to the velocity field. From the fact that the Lagrangian Λ_{f} is invariant with respect to local $SO(3)$ gauge transformations as well, we infer that a gauge field may be already known in fluid dynamics.

It was noted in the introduction that the gauge field Ω of a dynamical system, such as in the model of a nuclear rotation (Fujikawa and Ui, 1986), is defined only for the derivative with respect to the time t. This means that the replacement in the present fluid flows would be of the form, $\mathrm{D}_t \to \nabla_t = \mathrm{D}_t + \Omega(x)$ where the derivative $\mathrm{D}_t\boldsymbol{v}$ defined by (4) denotes the material derivative of a potential flow. In fact, we have $\partial_i(v^2/2) = v^k \partial_i v^k = (\partial_k \phi)\partial_i \partial_k \phi = (\partial_k \phi)\partial_k \partial_i \phi = v^k \partial_k v^i$.

According to the scenario of the gauge theory (e.g. Quigg, 1983) the velocity field $\boldsymbol{v}(x)$ and the covariant derivative $\nabla_t \boldsymbol{v}$ should obey the following transformation laws:

$$\boldsymbol{v} \mapsto \boldsymbol{v}' = \exp[\theta(t,x)]\boldsymbol{v}, \tag{6}$$

$$\nabla_t \boldsymbol{v} \mapsto \nabla'_t \boldsymbol{v}' = \exp[\theta(t,x)]\nabla_t \boldsymbol{v}, \tag{7}$$

where $R = \exp[\theta(t,x)]$ denotes a time-dependent local gauge transformation, and the covariant derivative is defined as

$$\nabla_t \boldsymbol{v} := \mathrm{D}_t \boldsymbol{v} + \Omega\boldsymbol{v} = \partial_t \boldsymbol{v} + \mathrm{grad}(v^2/2) + \hat{\Omega} \times \boldsymbol{v} \tag{8}$$

and θ, $\Omega \in \mathbf{so}(3)$,[4] where $\hat{\Omega}$ is the axial vector counterpart of Ω. The third term is determined so as to be compatible with the gauge transformation and Galilei transformation. From the above equations (6)–(8), it is found that the gauge field operator Ω is transformed to $\Omega' = \mathrm{e}^{\theta}\Omega\,\mathrm{e}^{-\theta} - (\partial_t\, \mathrm{e}^{\theta})\,\mathrm{e}^{-\theta}$. For an infinitesimal transformation $|\theta| \ll 1$, we have $\mathrm{e}^{\theta} = 1 + \theta + O(|\theta|^2)$. Using $\delta\theta$ instead of θ, the gauge field $\hat{\Omega}$ (in vector form) is transformed as

$$\hat{\Omega} \to \hat{\Omega}' = \hat{\Omega} + \delta\hat{\theta} \times \hat{\Omega} - \partial_t(\delta\hat{\theta}), \tag{9}$$

[4] The property θ, $\Omega \in \mathbf{so}(3)$ means that we are considering the principal fiber bundle.

up to the first order. The second term on the right-hand side describes the non-Abelian transformation law such as that of the Yang-Mills gauge field (Quigg, 1983, Chapter 4).

In addition, the covariant derivative $\nabla_t \boldsymbol{v}$ is required to be invariant with respect to the Galilei transformation $\boldsymbol{v}(x) \to \boldsymbol{v}_*(x) = \boldsymbol{v} - \boldsymbol{U}$ with $\boldsymbol{U}$ a constant vector. This is satisfied by

$$\hat{\Omega} = \mathrm{curl}\, \boldsymbol{v}, \quad \hat{\Omega}_* = \hat{\Omega}. \tag{10}$$

Thus, it is found that *the vorticity* $\boldsymbol{\omega} = \mathrm{curl}\, \boldsymbol{v}$ is the gauge field, and that the covariant derivative $\nabla_t \boldsymbol{v}$ is given by

$$\nabla_t \boldsymbol{v} = \partial_t \boldsymbol{v} + \mathrm{grad}(\tfrac{1}{2}v^2) + \boldsymbol{\omega} \times \boldsymbol{v} = \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \mathrm{grad})\boldsymbol{v}, \tag{11}$$

which is usually called the Lagrange derivative, or material derivative of $\boldsymbol{v}$.

According to the gauge principle, the derivative $\mathrm{D}_t \boldsymbol{v} = \partial_t \boldsymbol{v} + \mathrm{grad}(\frac{1}{2}v^2)$ of (3) should be replaced by the covariant derivative $\nabla_t \boldsymbol{v}$. Thus, we obtain the equation, $\nabla_t \boldsymbol{v} = -\mathrm{grad}\, h$. This is the Euler's equation of motion. In fact, using (11) for $\nabla_t \boldsymbol{v}$, we have

$$\partial_t \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{v} + \mathrm{grad}(\tfrac{1}{2}v^2) = -\mathrm{grad}\, h. \tag{12}$$

Equivalently, using the last expression of (11) and $\mathrm{grad}\, h = (1/\rho)\mathrm{grad}\, p$, we obtain $\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \mathrm{grad})\, \boldsymbol{v} = -(1/\rho)\, \mathrm{grad}\, p$. This is to be supplemented with the continuity equation,

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{v}) = 0. \tag{13}$$

4. Conclusion

According to the gauge principle, it is found that the gauge field coincides with the vorticity $\boldsymbol{\omega}$, and in addition, using the gauge-covariant derivative $\nabla_t \boldsymbol{v}$, the Euler's equation of motion (12) is derived for a homentropic flow. Taking curl of (12), we obtain the vorticity equation,

$$\partial_t \boldsymbol{\omega} + \mathrm{curl}(\boldsymbol{\omega} \times \boldsymbol{v}) = 0. \tag{14}$$

Using the continuity equation (13), this is rewritten as $(\mathrm{d}/\mathrm{d}t)(\boldsymbol{\omega}/\rho) = ((\boldsymbol{\omega}/\rho) \cdot \mathrm{grad})\boldsymbol{v}$. where $\mathrm{d}/\mathrm{d}t = \partial_t + (\boldsymbol{v} \cdot \mathrm{grad})$. Importance of material variation taking into account the motion of individual particles was pointed out by Eckert (1960) and Bretherton (1970). However, no consideration is given there on the *local gauge symmetry* in such a way as done in the present study, and the derivative such as (11) is assumed as an indentity in Bretherton. Because of the Clebsch representation in Eckert, the formulation is valid only locally.

There are some byproducts from the present formulation. The Noether's conservation law associated with the global $SO(3)$ gauge invariance is found to be the conservation of total angular momentum. In addition, the Lagrangian has a symmetry with respect to particle permutation, which leads to a local law of vorticity conservation, resulting in the vorticity equation as well as the Kelvin's circulation theorem. Therefore, the above Eq. (14) is a local conservation equation associated with a symmetry of particle permutation. Thus, it is found that the well-known equations in fluid dynamics are related to certain symmetries of the fluid system. The details will be discussed in a full paper (Kambe, 2003), where it is verified that Hamilton's principle together with isentropic material variations and the gauge-covariant derivative $\nabla_t \boldsymbol{v}$ leads to the Euler's equation of motion (12).

The present formulation provides a gauge-theoretical ground for physical analogy between the aeroacoustic phenomena associated with vortices (Kambe, 1986; Kambe and Minota, 1987) and the electron and magnetic-field interactions. In particular, there is a close analogy between the Aharonov-Bohm effect (Berry et al., 1980; Peshkin and Tonomura, 1989) and the scattering of a sound (or water) wave by a rectilinear vortex (Kambe and Mya Oo, 1981; Coste et al., 1999).

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Appendix A. Galilei transformation of velocity field

The Galilei transformation is considered to be a limiting transformation of the Lorentz transformation of space–time $(x^\mu) = (t, x)$ as $v/c \to 0$. The Lorenz covariant Lagrangian $\Lambda_{\mathrm{L}}^{(0)}$ in the limit as $v/c \to 0$, is defined by

$$\Lambda_{\mathrm{L}}^{(0)}\,\mathrm{d}t = \int_M \mathrm{d}^3x\,\rho(x)\left(\frac{1}{2}\langle v(x), v(x)\rangle - \varepsilon - c^2\right)\mathrm{d}t \tag{A.1}$$

(Landau and Lifshitz, 1987, Section 133). The third $-c^2\,\mathrm{d}t$ term is not only necessary, but indispensable, so as to satisfy the Lorenz-invariance (Landau and Lifshitz, 1975, Section 87). This term gives a constant $c^2\mathscr{M}\,\mathrm{d}t$ to $\Lambda_{\mathrm{L}}^{(0)}\,\mathrm{d}t$, where $\mathscr{M} = \int \mathrm{d}^3x\,\rho(x)$ is the total mass in the flow domain. In carrying out variation of $\mathscr{A}$, the total mass $\mathscr{M}$ is fixed to a constant. In the present analysis, corresponding Lagrangian of fluid motion in the Galilei system is given by the first integral of (1). In this Lagrangian Λ_{f}, local conservation of mass is imposed. Therefore the mass is conserved globally as a consequence of local conservation. Only when we need to consider Galilei invariance, we use the Lagrangian $\Lambda_{\mathrm{L}}^{(0)}$.[5]

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[5] We have $\langle \boldsymbol{v}, \boldsymbol{v}\rangle - \langle \boldsymbol{v}_*, \boldsymbol{v}_*\rangle = 2\langle \boldsymbol{v}, \boldsymbol{U}\rangle + \frac{1}{2}U^2 = (\mathrm{d}/\mathrm{d}t)(\langle x(t), 2\boldsymbol{U}\rangle + \frac{1}{2}U^2 t)$. Within the frame of Newtonian mechanics, the total time derivative term is understood not to play any role in the variational formulation.

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